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Weyl on Fregean implicit definitions: between phenomenology and symbolic construction

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Abstract

This paper aims to investigate certain aspects of Weyl's account of implicit definitions. The paper takes under consideration Weyl's approach to a certain kind of implicit definitions i.e. abstraction principles introduced by Frege. Abstraction principles are bi-conditionals that transform certain equivalence relations into identity statements, defining thereby mathematical terms in an implicit way. The paper compares the analytic reading of implicit definitions offered by the Neo-Fregean program with Weyl's account which has phenomenological leanings. The paper suggests that Weyl's account should be construed as putting emphasis on intentionality of human mind towards certain *invariant* features of the elements of initial domains of discourse that are involved in equivalence relations. Definition of terms like *direction*, *shape*, *number* etc. is achieved by a kind of transformation of those invariants into *ideal* objects that is involved in intuition. Then the paper argues that at the period (1926) of Weyl's writings on implicit definitions, he is inclined to endorse symbolic construction as a way to explicate the objectivity of certain processes as those that are carried out in case of implicit definitions.

Key words: definition, equivalence relation, intentionality, invariant, intuition, ideal object

1. Introduction

Weyl offers an account of implicit definitions in his (1926) *Philosophie der Mathematik und Naturwissenschaft* (expanded and translated as *Philosophy of Mathematics and Natural Science*, Princeton University Press, 1949, and recently reprinted by Princeton University Press, 2009)[1]. Most of the examples he has taken under consideration concern Fregean contextual definitions about which

[1] Citations of (1926) in the text follow the 2009 publication.

much philosophical discussion has been recently made in the context of the Neo-Fregean[2] program. Yet, Weyl's approach is very different since it construes implicit definitions in a way that does not avoid to take seriously in account his phenomenological leanings. Weyl's work has passed through several phases. In *Das Kontinuum* (1918) Weyl was strongly influenced by Husserl's philosophy of mathematics and logic. Some years after the publication of it, he was impressed by Brower's intuitionism and his intuitionist phase lasted until 1924. He then admitted that intuitionistic mathematics could not account for mathematics used in natural science so he changed his philosophical position by making justice to Hilbert's program. Further, as Mancosu and Ryckman (2002) maintain, Weyl was inclined to do justice to formalism at the decade of twenties, so departing both from intuitionism and phenomenological tradition. This paper argues that Weyl's elaboration of implicit definitions in (1926) shows off, on the one hand his phenomenological leanings, on the other hand an inclination towards symbolic mathematics that comes to overstep the intuitively accessible.

Particularly, the paper focuses its attention on the way Weyl construes the procedure by which certain mathematical terms e.g. *shape*, *direction*, *congruent integer* etc. are implicitly defined. It compares his account with the Neo-Fregean one and pinpoints the differences. It argues that Weyl retains certain phenomenological commitments with regard to the topic of mathematical definition at the time of his 1926 work, but he endorses, at the same time, the symbolic reconstruction as a necessary way to go beyond the intuitive.

2. Abstraction Principles as implicit definitions

Frege presented abstraction principles in *Grundlagen der Arithmetic* (1884) to introduce certain sortal concepts, e.g. of natural number, direction, shape, etc. Abstraction principles are taken to be implicit definitions in the recent philosophical literature concerning the Neo-Fregean program and the discussion about it. An implicit definition, in general, is a stipulation of a sentence (or a set of sentences) as true that contains (contain) the *definiendum* (*definienda*). Implicit definitions do not aim to interpret the *definienda* into already known expressions as explicit definitions do. They are not definitions of the form '$=_{df}$' hence, they do not eliminate the content of the *definiendum* in favour of the *definiens*, as e.g. in case of 'bachelor $=_{df}$ an unmarried adult man'. Instead, implicit definitions fix some pattern of use of the *definienda* by stipulating certain sentences (or sets of sentences) as true. In particular, abstraction principles have been construed as implicit definitions too. They are the outcome of the stipulation of appropriate bi-conditionals as true which determine truth conditions for sentential contexts in which the *definienda* appear. So they fix patterns of use for the *definienda*. (cf. MacBride 2003; Hale & Wright 2001; Hale 2001). Some examples follow:

In *Grundlagen*, §§64-68, Frege introduced the principle (D=) that fixes truth conditions for sentential contexts of the form "D(*a*) = D(*b*)" ("The direction of line *a* is the same as the direction of line *b*") in which the *definienda* (the singular terms 'D(*a*)', 'D(*b*)') occur.

(D=) $(\forall a)(\forall b)\ [(D(a) = D(b)) \leftrightarrow (a // b)]$

[2] The Neo-Fregean program (Neo-Logicism, Neo-Fregeanism) offered a systematic analytic defense of Frege's two basic claims according to which arithmetic is reduced to logic and natural numbers are abstract objects.

In this case, *a*,*b* denote straight lines, i.e. the variables bound by the quantifiers take their values from an initial domain of straight lines.
(:*The direction of the line a is the same as the direction of the line b if and only if a is parallel to b*)[3].
The Principle (D=) is taken to implicitly define the concept of *direction* of a line by establishing a pattern of use for the terms 'D(*a*)', 'D(*b*)' ('the direction of *a*', 'the direction of *b*') (cf. Hale & Wright 2001 131).

Frege introduced the principle (N=) too (or, as it is otherwise been called, Hume's Principle[4]) that fixes truth conditions of sentential contexts of the form "Nx:Fx=Nx:Gx" ("The number of the concept *F* is the same as the number of the concept *G*") in which the *definienda* (the terms 'Nx:Fx', 'Nx:Gx') occur.

$$(N =) \qquad (\forall F)(\forall G)\ [(Nx{:}Fx = Nx{:}Gx) \ \leftrightarrow\ (F\ 1 - 1\ G)]$$

Here, *F*, *G* denote concepts, i.e. the variables bound by the quantifiers take their values from a domain of concepts. This is why the abstraction principle in question is second order quantified.
(*The number of the concept F is the same as the number of the concept G if and only if (the instances of) F and G are 1-1 correlated*).
An example: let *F* be the concept *guest* and *G* be the concept *dinner plate* so the number of the guests is the same as the number of dinner plates if and only if the instances of the two concepts are 1-1 correlated.

D= and N= (HP) are universally quantified bi-conditionals that establish truth conditions of appropriate contexts (: identity sentential contexts) in which the *definienda* occur. The *definienda* acquire their meanings in such contexts according to Frege's context principle: the meaning of an expression should always be defined in the context of a whole sentence (it cannot be defined independently of a sentential context). Frege regarded his abstraction principles as contextual definitions. Yet, he was very disappointed when he found out a characteristic of them that is related to the so called 'Ceasar's problem' (*Grundlagen*, §66). He noticed that a sentential identity context like "j = Nx:Gx" cannot be paraphrased as the right hand side of the bi-conditional of HP indicates. For example, let j be the name 'Julius Ceasar'. Then the context "Julius Ceasar = Nx:Gx" cannot be paraphrased as the right hand side of HP indicates, i.e. as an 1-1 correlation among instances of two concepts. This fact stroke Frege as a serious difficulty of his abstraction principles to function as satisfactory definitions. In fact, the abstraction principle does not answer the question whether an object of the world that we pick up is a number or not. This is why Frege abandoned the attempt to define directions and numbers by means of contextual definitions and he went on to define them by giving explicit definitions through extensions[5]. The problem Frege came up against was that his abstraction principles are not explicit definitions since they do not provide for the elimination of their *definienda* in all contexts of occurrences. Crispin Wright (1983) clarified the same fact by explaining

[3] For this formulation in ordinary language cf. *Grundlagen* §64.
[4] The name is due to the fact that when Frege introduced the principle in question, he reminded of Hume's considerations about numerical identity.
[5] For Frege's *explicit* definitions cf. *Grundlagen* §68: "the direction of the line *a* is the extension of the concept 'parallel to the line *a*'" and "the number of the concept *F* is the extension of the concept 'equinumerous to the concept *F*'".

that an abstraction principle (e.g. HP) cannot eliminate the *definienda* (e.g. the terms 'Nx:Fx', 'Nx:Gx') in all contexts, in favour of the *definiens* (: the 1-1 correlation between the instances of two concepts) (cf. Wright 1983 135-136; Hale & Wright 2001 12). An identity context "j = Nx:Gx" can be paraphrased as an 1-1 correlation between the instances of two concepts only in case that 'j' has the same form ('Nx:Fx') with the second term that appears in the identity context. So abstraction principles are not *explicit* definitions. After Wright's remark, abstraction principles are taken to *implicitly* define the singular terms which occur on the left hand side of the relative bi-conditional by fixing the truth conditions of an identity sentential context and establishing a certain pattern of use of the *definienda* (cf. Hale & Wright 2001 12-14 142-150; MacBride 2003 110-112)[6].

Abstraction principles, systems of axioms and Carnap conditionals[7] are three basic kinds of implicit definitions under investigation nowadays (cf. Hale & Wright 2001 117-150). In case of axiomatic systems, a set of sentences are stipulated as true in which the *definienda* appear. The set of sentences in question define more than one terms ('*f*', '*g*', '*h*' etc.) simultaneously. For example, on the basis of Hilbert's axiomatization, the system of Euclidean axioms define collectively the geometrical terms 'point', 'line', 'plane' by establishing certain mutual relations that points, lines and planes should satisfy. Analogously, the system of Peano axioms define implicitly 'number', 'zero', 'successor' on the basis of mutual relations and conditions they should satisfy. The Frege – Hilbert controversy about whether axioms can define mathematical terms is well known (cf. Resnik 1974). Frege rejected the use of axiomatic systems as definitions of any kind but in Hilbert's sense, systems of axioms are implicit definitions. However, it has been ironically tragic that although Frege did not like implicit definitions, his own abstraction principles are construed as implicit definitions too in the recent discussion (not of course, in the way that systems of axioms are taken to be but according to the account previously presented).

As previously noted, abstraction principles are stipulations of appropriate bi-conditionals that settle truth conditions of identity sentential contexts in which the *definienda* occur. I will present MacBride's (2003 110-111) description of the way an abstraction principle functions as an implicit definition in the general case. The general form of an abstraction principle is the following and it works in the following way:

(Σ=) $(\forall a)(\forall b)\ [(\Sigma(a) = \Sigma(b)) \leftrightarrow (a \approx b)]$

Let 'a', 'b' be terms of a language that is already in use and '$\approx$' a relational predicate that expresses an equivalence relation R among the elements a and b of an initial domain (R: a relation that is reflexive, symmetrical and transitive). The bi-conditional in question is stipulated as true. It extends the already familiar language by introducing an operator Σ which produces new singular terms '$\Sigma(a)$', '$\Sigma(b)$', ... e.t.c. (For example, in case of HP, the arithmetical operator N is the operator that extends the initial language by producing the singular terms 'Nx:Fx', 'Nx:Gx',...). Moreover, the bi-conditional

[6] Particularly, in their paper (2001 144-146) Hale & Wright argue that HP is an implicit definition that avoids *arrogance*. A more general suggestion is made that conditional in form implicit definitions (like abstraction principles) avoid arrogance.

[7] A Carnap conditional has the form "$\exists x(\#x) \rightarrow \#f$" (: if something satisfies such and such conditions then f satisfies those conditions). It is taken to be an implicit definition of a theoretical scientific term f, e.g. 'electron'. (cf. Hale & Wright (2001); Psillos & Christopoulou (2009))

establishes truth conditions for the novel equality-contexts of the form '$\Sigma(a)=\Sigma(b)$'. So we set a definition that *implicitly* accrues meanings to the term-forming operator (Σ) and the newly formed terms ('$\Sigma(a)$', '$\Sigma(b)$', e.t.c.) (cf. Hale & Wright 2001 142). The abstraction principle Σ= is not a definition of the form '$=_{df}$' that is, it does not eliminate the content of the *definienda* in favour of the *definiens* hence it is not an *explicit* definition. Instead, it fixes some pattern of use of the *definienda* by determining truth conditions for sentential contexts in which the *definienda* appear. On the account presented, abstraction principles are taken to be implicit definitions. Further, the discussion concerning abstraction principles as implicit definitions includes many issues about certain properties that abstraction principles should possess, e.g. consistency, non-arrogance, conservativeness, etc in order to work as good implicit definitions. For example, HP is taken to possess all the above virtues.[8]

I will remind of some main points of the NeoFregean account of abstraction principles. An abstraction principle accomplishes two main tasks from which the first is concept formation. According to the advocates of Neo-Fregeanism (Hale & Wright 2001 117-150), abstraction principles introduce certain concepts (e.g. the concept of *direction*, the concept of *number*, the concept of *shape* e.t.c.). Recall that Frege himself maintained that certain concepts (e.g. the concept of *direction*, the concept of *number*) are obtained by a process of *recarving the content* of the right-hand side of the relative bi-conditional in a *new* way on the left-hand side. In *Grundlagen* §64 he holds, for example, that the concept of the *direction* of a line is introduced by *recarving the content* of parallelism of straight lines (the right-hand side of D=) in a new way on the left-hand side as an identity between *directions* of those lines. Further, Frege regards the right-hand side of D= as quite familiar to our cognitive equipment (more familiar than the left) since parallelism itself is given to us in intuition. Similarly, the concept of natural number is introduced (in case of Hume's Principle) by recarving the content of 1-1 correlation among the instances of concepts *F* and *G* (right-hand side of *Hume's Principle*) in a new way on the left hand side as an arithmetical identity. Of course, 1-1 correlation is quite familiar to our cognitive equipment not because it is given in intuition as in case of parallelism but because it is a (second-order) logical relation. Recall that Frege agrees with Kant that geometrical knowledge arises from intuition however he holds that arithmetical knowledge origins from logic.

The question that arises is what does Frege mean by the notion '*recarving the content*'? In fact, he suggests that identity statements (statements of direction and statements of numerical identity) can be viewed as carving up in a new way the content of statements asserting the relevant equivalence

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[8] *Non – arrogance* is a property of an implicit definition. An implicit definition should be *non-arrogant* i.e. it should not need any a posteriori work for its affirmation. For example, according to Hale & Wright (2001 142-150), HP avoids *arrogance* because of its (double) conditional form. Besides, an implicit definition is *conservative* if and only if it does not imply new consequences that were not already implied by the initial language before the very stipulation takes place. In fact, HP does not imply new consequences that were not already implied before the extension of the initial vocabulary takes place by means of the introduction of the operator N. Some abstraction principles are consistent, conservative, non arrogant and work successfully as implicit definitions. The discussion about the virtues of abstraction principles that work as good implicit definitions is broad as well as rich. (For example, cf. discussion about the so called "bad company" i.e. a class of abstraction principles that lack certain virtues of good implicit definitions (Linnebo 2009; Eklund 2009, etc.)

relations (parallelism, 1-1 correlation) among lines and concepts respectively. A very interesting account of that crucial notion has been provided by the advocates of Neo-Fregeanism, especially Hale (2001). Hale (2001 341) notes that *content* is difficult to be construed since Frege soon after *Grundlagen* replaced it by the notions of thought and truth value (as special case of sense and reference distinction). However, Hale suggests that *content* should be regarded as a *weak notion* of *sense*. He explains that the two sides of an abstraction principle coincide in *weak sense* if and only if they share the same truth conditions. For example, the identity of directions and parallelism of lines that appear on the left hand side and the right hand side of D= respectively share the same truth conditions. Further the standard interpretation of the notion of *recarving the content* by the NeoFregeans is based on their notion of *reconceptualization*. They suggest that the left hand side of an abstraction principle is no more than reconceptualization of the states of affairs depicted by the corresponding right hand side. Hence, an abstraction principle is taken to *reconceptualise* the states of affairs that are described by the right-hand side of its bi-conditional. The states of affairs described by the right hand side of D= are given to us as the obtaining of an equivalence relation (parallelism) among lines so we have the option of reconceiving such states of affairs as an identity of a new kind of thing, i.e. directions. The very fact that two lines are parallel constitutes the identity of their directions (Hale and Wright 2001 277). So, *reconceptualization* of the states of affairs (parallelism) that are described by the right-hand side of the bi-conditional D= in a new way provides a new concept: the concept of *direction* of a line. Analogously, *reconceptualization* of the states of affairs (1-1 correlation) that are described by the right-hand side of *Hume's Principe* provides an identity of natural numbers. Concepts whose instances are 1-1 correlated (e.g. concept *wheel of my car* and concept *leg of my desk*) possess the same number (four).

There is a second task that an abstraction principle accomplishes: it makes us able to recognize certain ontologies of abstract objects, e.g. *directions*, *numbers*, e.t.c. The abstract objects in question are the very instances of the concepts that are introduced by means of abstraction principles. For example, *Hume's Principle* defines (implicitly) the concept of *natural number* whereas each individual natural number is an instance of that very concept. The instances of the concept *direction* are abstract objects etc. The abstract objects in question are the objects of reference of relative singular terms. Hence, the number 1 is the object of reference of the singular term 'the number of the concept *F*' whereas *F* may be the concept *moon of the earth*.

3. Hermann Weyl on definition by abstraction

Weyl offers an example that arises from an axiom about circles in order to show the way creative definitions work. According to it, 'a point O and a different point A determine a circle, the "circle about O through A"; that a point P lies on this circle means that OA=OP.' This axiom on its own defines implicitly a circle. Yet, Weyl emphasizes the fact that a circle is given *only* by a point O, a (different) point A and what is meant by 'the point P lies on the circle'. Then he lays down truth conditions for circles to be identical. In principle, the criterion of identity of circles prescribes that the circle about O through A is identical with the circle about O′ through A′ if and only if all points on the first circle also lie on the second circle. However, Weyl avoids involvement of infinite manifolds of points by replacing this criterion by the criterion (C): 'the circle about O through A is identical with the circle about O′ through A′ if and only if O coincides with O′ and OA=O′A′'. Weyl presents it as an example of *creative* definition and he stresses the point that on such a setting any reference to infinite manifold is ruled out. This is important because he endorses the view that mind can carry out only finite creative acts and infinite as closed and complete in itself is rejected. By the way, the rejection of the infinite as closed and complete is a characteristic of intuitionism as well as the phenomenological tradition. In Weyl's view, to construct definitely a circle, one needs only two different points O and A, and understanding what is meant by 'the point P lies on the circle'. The latter is understood by the equality OP=OA.

The above act is constructivist (*creative* in Weyl's terms). Yet, functional expressions are useful in order to present it. Weyl takes 'the circle about O through A' to be the value of a function Φ of O and A as the arguments. Then the above criterion stated as (C) says that the value of the function Φ of the points O and A is the same as the value of the function Φ of the points O′ and A′ if and only if O coincides with O′ and A coincides with A′. According to this setting however, the criterion (C) can perfectly be regarded as an abstraction principle. Despite the fact that Weyl does not mention it, it is clear that he defines circles by means of an abstraction principle.

Hermann Weyl mentions Frege as the philosopher who formulated the method of definition by abstraction principles in all generality and he remarks that Helmholtz elaborated it too in the 19^{th} century.[9] Weyl (1926 8-10) asserts that abstraction principles origin from equivalence relations among elements of an initial domain D. He describes the procedure in question by exposing certain examples. The first one he takes under consideration is a principle based on a relation of *similarity* among geometrical figures. Geometrical similarity $\approx_s$ is an equivalence relation (reflexive, symmetrical, transitive) so Weyl puts down the bi-conditional: "the *shape* of the figure g is the same as the *shape* of the figure h if and only if the figure g is *similar* to the figure h". This can be written down by the standard form of an abstraction principle:

$$(S=) \quad (\forall g)(\forall h)\ [(S\,(g) = S(h)) \leftrightarrow (g \approx_s h)]$$

(In other words, two figures have the same shape if and only if they are geometrically *similar* to each other). It should be remarked that the abstraction principle (S=) was firstly introduced by Frege in *Grundlagen* §64, many years before Weyl's writings, as a definition of the concept of *shape* of a triangle (two triangles have the same shape if and only if they are *similar* to each other).

A second example of abstraction that Weyl puts forward (that is not introduced by Frege) is about integers which are *congruent modulo m*. Weyl writes: "two integers according to Gauss are *congruent modulo* 5 if their difference is divisible by 5" (1926 10). In the general case, two integers x and y are *congruent modulo m* if and only if their difference is divisible by *m*. Weyl believes that one can obtain the *congruence-integers modulo m* from the domain Z of integers by means of an abstraction principle. To reconstruct Weyl's idea, suppose we lay down the following equivalence

[9] Weyl cites Frege, *Die Grundlagen der Arithmetik,* Breslau 1884, §§ 63-68 as well as Helmholtz, 1887, Zählen und Messen, 1887, *Wissenschaftliche Abhandlungen*, III, p. 377.

relation R : x-y is divisible by 5 (i.e. x-y = 5k, k integer).[10] Then we obtain : $x \equiv y$ (*mod* 5) (i.e. x and y are *congruent mod 5*).

So, let a principle stating that

"x and y are *congruent mod 5* if and only if their difference is divisible by 5".[11]

$$(\forall x)(\forall y)\ [x \equiv y\ (mod\ 5) \leftrightarrow x\text{-}y \text{ is divisible by } 5] \qquad x, y \text{ integers}$$

This principle can take the more general form:

$$(\forall x)(\forall y)\ [x \equiv y\ (mod\ m) \leftrightarrow x\text{-}y \text{ is divisible by } m] \qquad x, y \text{ integers}$$

m integer, $m>1$

(x and y are *congruent mod m* if and only if their difference is divisible by m)

However, since all abstraction principles make use of an operator that correlates elements of the initial domain D to singular terms, we need here an operator T_m in order to give the above principle the standard form of a bi-conditional of an abstraction principle:

$$(C=) \qquad (\forall x)(\forall y)\ [T_m(x) = T_m(y) \leftrightarrow x\text{-}y \text{ is divisible by } m]$$

(According to this formulation, $T_m(x)$ and $T_m(y)$ are the values of the operator T_m for x and y as the arguments. The left hand side includes an identity statement whereas the right hand side includes the initial equivalence relation R between x and y stated above).

The focal point, according to Weyl, is that one has to take under consideration the *invariant* features of the elements x and y that are correlated by the equivalence principle R. In case of the example S= mentioned above, similarity of figures and equality of the angles of similar figures express certain features that remain invariant. In case of the second example, i.e. principle C=, invariance is expressed by the fact that x and y leave the same remainder if divided by m. In order to explain what really happens, one should think that the equivalence relation R: 'x-y is divisible by m' forms a partition of the domain of integers Z that yields the equivalence classes: C_0, C_1, ... C_{m-1}. Each one of those classes includes integers which are *congruent mod m* and if, divided by m, they leave the same remainder. So the members of the above classes leave remainder 0, 1, … m-1 respectively. Weyl remarks that the result of the (above) abstraction procedure C= is a finite domain of 5 elements (in the general case: m elements) that is obtained from an initial infinite domain of integers. The finite domain is: $\{C_0, C_1, ... C_4\}$ (in the general case: $\{C_0, C_1, ... C_{m-1}\}$). He further stresses the important point (well known by the number theorists) that the operations of addition and multiplication[12] are invariant with respect to congruence, so the usual algebra can be carried on as well as in the infinite domain Z of the ordinary integers (cf. 1926 10).

Weyl's account of the procedure has the following main characteristics:

a. Weyl holds that elements of the original domain, that is: figures, lines, integers etc. that are equivalently related by an equivalence relation R have *a feature in common* that remains *invariant*

[10] It is easy to see that this relation is reflexive, symmetrical and transitive.

[11] Equivalently, x and y are *congruent mod 5* if and only if, divided by 5, they leave the same remainder.

[12] Addition: $T_m(x) + T_m(y) = T_m(x+y)$, Multiplication: $T_m(x) * T_m(y) = T_m(x*y)$, x, y, m integers, m>1, It can be proved that the set $\{ C_0, C_1, ... C_{m-1} \}$ is an abelian group with regard to addition and a reversing semi-group with an identity element with regard to multiplication. Besides, multiplication is distributive to addition.

through the procedure. This is obvious in all cases we have come across so far, for example, in case of the abstraction principle S=, the equivalence relation *R* in question is *geometrical similarity*, whereas two figures that are taken to be geometrically *similar* to each other have the same *shape* (: a feature in common). In case of the abstraction principle D=, the equivalence relation *R* in question is parallelism whereas lines that are taken to be parallel to each other have the same *direction* (: a feature in common) etc.

b. An important point in Weyl's account (cf. 1926 11) is that the alleged *feature in common* previously mentioned is being transformed into an *ideal* object by means of an abstraction principle. For example, a *shape* that is the common feature of two geometrically similar figures is transformed into an *ideal* object. Analogously, Weyl thinks that the *colour* of a flower is a *feature in common* of two or more particular flowers and that can be transformed into an *ideal* object. *Ideal* objects arise as the outcome of transformation of the common features of things that are involved in the respective abstraction principles as equivalent by an equivalence relation *R*.

c. Weyl's account departs from Frege and is different also from the Neo-Fregean approach so long as it endorses a kind of constructivism. I will recall the point that according to Frege, numbers are abstract objects (cf. *Grundlagen*, §62). Besides, according to the Neo-Fregean approach, *shapes*, *numbers*, *directions* etc. are abstract objects, hence, not creations of human mind. Wright (2001 278) characteristically notes that it is important to be clear that it would be a misconception to view this task of an abstraction as involving the notion that abstract objects are creations of the human mind. He stresses the point that what is formed –created– by such an implicit definition is a concept e.g. the concept of number (not the abstract objects that are the instances of that concept e.g. the particular numbers). The existence of the alleged abstract objects results from the truth of certain statements (Wright 1983 148). On the other hand, Weyl construes abstraction principles in constructivist lines, hence his definitions are creative in some sense that, however, will be clarified later, in sections 4 and 5.

d. *Intuition* has an important role in Weyl's account of implicit definitions however Frege's account takes intuition to be important only in case of geometrical examples at the starting point of the procedure. For example, in case of the abstraction principle D= concerning directions of lines, Frege holds that parallelism (the equivalence relation between lines) from which the abstraction principle origins, is given to us in intuition (*Grundlagen* §64). By contrast, in case of arithmetic Frege departs from Kant and supports the view that arithmetic is founded on logic alone. His arithmetical example of abstraction principle N= (Hume's Principle) mentioned in the first section of this paper is based on an equivalence relation (1-1 correspondence) that is taken to hold in (2^{nd} order) logic alone.

4. The role of intention and intuition

As noted in section 2, there is a crucial stage in the procedure which Frege puts on as *recarving the content* of the right hand side of an abstraction principle (the equivalence relation among certain elements of an initial domain D) and reformulating it as an identity statement on the left hand side. This process is construed in terms of *reconceptualization* in the Neo-Fregean context as we saw. However, Weyl (1926 11) construes the procedure in question in the context of an entirely different philosophical tradition. On his account, there is also a kind of transformation but he puts it differently as following: the procedure discloses the alleged *common features* of certain items of the original domain D that are correlated by equivalence relations. Further a transformation of those *common features* into *ideal* objects takes place. The role of intuition is decisive in that process of transformation. I will attempt to reconstruct the process in question according to the points a. and b. made in the previous section.

Weyl was influenced by Husserl so it is plausible that he puts emphasis on the role of mind in structuring and forming *ideal* objects in phenomenological terms. In the first place, he attributes a special role to intentionality by stating that the mathematician has an '*intention to consider exclusively certain invariant properties and relations among the originally given objects*' (1926 9). The notion of *intentionality* is endorsed to describe directedness of certain mental acts like thinking, believing, e.t.c. to certain invariant properties and relations that were discussed in section 3 with regard to his examples of abstraction principles S= and C=. Recall that in those cases, we came across such invariant characteristics like *similarity* of figures (e.g. triangles) whose corresponding sides have invariant ratios and their relative angles have the same numerical values or, *congruent* integers mod *m* that leave the same remainder if divided by *m*. Moreover, in case of congruent integers mod *m*, it was noted that addition and multiplication are invariant with respect to *congruence* mod *m*. Hence in all cases of abstraction principles, certain invariant properties of the elements of the original domain constitute the starting point to which the act of thinking is directed. In fact, the equivalence relation from which an abstraction procedure origins, provides the *invariant* elements towards which the human mind is intended. Yet those invariant properties are exposed by means of certain features that the elements of the original domain have in common. Recall that in section 3, we saw that according to Weyl, lines, integers etc. that are equivalently related by an equivalence relation *R* have *a feature in common* in each case. For example, two geometrically *similar* triangles have a *shape in common* whereas two parallel lines have a *direction in common*, etc. To stress the point concerning the common features in question, Weyl remarks that two flowers may have a common feature (colour 'red') and "*the general procedure of constructing these remainders and these numerical values of angles and ratios, respectively, (in the mathematical cases) takes the place of the feature 'colour', its identical result for two integers or triangles that of the identical 'red' of two flowers*" (1926 11). However, the next step after directedness is transformation of those features into *ideal* objects.

Generation of *ideal* objects is the second creative act in the procedure under consideration which shows off Weyl's phenomenological leanings. In the phenomenological tradition the role of intuition is quite important in the constitution of *ideal* objects. An object is constituted if it is brought into light by a certain process that is involved in intuition hence, intuition enables the object to disclose itself and become present to mind (cf. Sokolowski 2000). The mind is directed towards various states of affairs, properties or relations that remain invariant but in order to obtain knowledge, the intentions need to be fulfilled (Husserl 1900/1[13]). That is, unless intuition fulfills an intention, the object in question is still absent (it is not present to mind). Hence, fulfillment of certain intentions towards invariant features and relations results in constitution of ideal objects (cf. Tieszen 2005). In the light of

[13] Investigations VI, §§ 1-29 offers an account of the issue of how an intention is fulfilled.

the above account of how ideal objects are constituted, we can explain the way by which ideal objects are generated by abstraction .

As we saw, at the beginning of the procedure of an abstraction principle, the mind is directed towards certain properties and relations, features of things of the original domains that remain invariant. However, the alleged intentions should be fulfilled by intuition in all those cases that were discussed so far. Thus, intuition involves a finite process that is carried out in time so that those features concerning the alleged invariants in question, e.g. *shapes* of figures, *directions* of lines, *congruent* intengers etc. come to disclose themselves and become present to mind as *ideal* objects Weyl remarks that *"…the transformation of a common feature into an ideal object… is an essential step*" (1926 11). [14]

Consequently, an abstraction principle involves a finite process of intuition that fulfills certain mathematical intentions and generates *ideal* objects. Existence of Weyl's *ideal* objects should be construed in a transcendental sense. Existence of *ideal* objects is given in intuition as the content of the processes of consciousness. However, as we saw, *ideal* objects that are generated by abstraction are taken to emerge from *invariant* properties and relations of things that are correlated by an equivalence relation *R*. So they are constituted in intuition as the outcome of a kind of transformation of certain invariant states of affairs.

5. Symbolic construction

The importance of the *symbolic* in Weyl's account of implicit definitions (like abstraction principles or the definition of a circle) mentioned above is another point that this paper aims to stress. It has been remarked (cf. Mancosu and Ryckman 2002) that during the decade of twenties, Weyl made justice to formalistic methods since he admitted that phenomenology as well as intuitionism were insufficient for the understanding of the whole mathematics since many parts of mathematical theories cannot be reduced to what is intuitively accessible. Weyl believed that a further step (beyond the intuitively accessible) is necessary in mathematical knowledge and this is constituted in symbolic construction. Hence, one can detect his inclination to symbolic construction in his (1926). He writes that "… *It took a long time for mathematics before it had acquired the constructive tools to cope with the complexity and variety of such intuitively given figures. But once it had reached that stage the superiority of its symbolic methods became evident*" He also concludes: "*all knowledge while it starts with intuitive description, tends toward symbolic construction*" (1926 75).

In regard with the topic of implicit definitions his inclination to symbolic construction is obvious too. Intuitive exhibition meets symbolic construction in case of abstraction principles. Weyl makes use of symbolic tools, like functions $\Phi(u, v, \dots)$ with one or more arguments that vary freely within certain domains. In particular, such functions with one blank $\Phi(u)$ are taken by him to present properties. However, he thinks that functions $\Phi(u, v)$ with two arguments may also be in use that present binary relations and so on. Such functions (operators) correlate elements of an initial domain to

[14] The intuitionist leanings of Weyl (which involve the importance of intuition too) are not adequate enough to explain Weyl's persistence on the *transformation of the invariant features into ideal objects*. The latter is a phenomenological notion. So it is the phenomenological leanings of Weyl that justify his claim about the *transformation* in question.

certain properties, relations etc., that is, to the alleged *features* in question we saw in the previous section. For example, if u and w are figures, Φ(u) is the *shape* of the figure u and Φ(w) is the *shape* of the figure w then an abstraction principle can be laid down according to which Φ(u) is identical to Φ(w) if and only if u is geometrically *similar* to w. However, in case of the definition of the circle, we need a binary function Φ(u, v). We saw that Weyl takes 'the circle about O through A' to be the value of a function Φ of O and A as the arguments. So Φ(O, A) is identical to Φ(O′, A′) (i.e. the value of the function Φ of the points O and A is identical to the value of the function Φ of the points O′ and A′) if and only if O coincides with O′ and A coincides with A′. Similarly, one can make use of ternary, quaternary relations etc.

It is worth mentioning that the Neo-Fregean account deals with symbolism in similar lines. Appropriate operators form functional expressions like 'the *shape* of the figure *g*', 'the direction of the line *a*' etc. Bob Hale (1987 35) maintains that the Fregean procedure of providing terms for directions and shapes is generalized to yield a functional singular term T(*a*). An abstraction principle is then formed by the following bi-conditional:

T(*x*) = T(*y*) if and only if *Rxy*

(The value of the operator T of *x* as the argument is identical to the value of the operator T of *y* as the argument)

So, the use of functions-operators by means of which functional expressions like 'the *shape* of the figure *g*' are formed is significant in both accounts (of Weyl and Frege) because it plays a crucial role in the process of the alleged transformation of equivalence relations of the right hand side of abstraction principles into identities occurring on the left hand side. However, we saw that there is a difference in that on the one hand, Weyl holds that the alleged functional expressions e.g. 'the *shape* of the figure *g*', e.t.c. stand for certain *features* (in common) (e.g. *shape, direction,* e.t.c.) that are purported to be transformed into *ideal* objects. On the other hand, according to the Fregean view, the functional expressions in question are singular terms that are purported to refer to abstract objects. Despite the differences that were discussed in previous sections, symbolic construction represents the process in a uniform way and fixes the standard form of abstraction principles as bi-conditionals (with an identity relation on the left-hand side and an equivalence relation on the right-hand side). Hence, it makes explicit the standard way all abstraction principles work.

It should be asserted that the symbolic is not a mere tool of communication in this phase of Weyl's writings. As he notes "*consciousness makes the attempt to jump over its own shadow, … to represent the transcendent- but, how could it be otherwise?, only through the symbol*" (1926 66). This point needs clarification. In the previous section we saw that Weyl's *ideal* objects are generated in intuition as the outcome of the processes of consciousness. What is at stake here is the requirement of objectivity that needs to be fulfilled. If ideal objects become present in intuition as the outcome of certain processes of an individual consciousness then what safeguards that processes carried out by individual consciousnesses conclude uniformly in the same outcomes?

However, the requirement of objectivity can be met in the following way. As I noted in sections 3 and 4 an important point of Weyl's approach concerns the *invariance* of certain features of the elements of the initial domain that remain stable through the process of an abstraction principle.

Symbolic expressions such as S(*g*) (:‘the shape of the figure *g*’), D(*a*) (:‘the direction of the line *a*’) etc. explicate those *invariants* and present them as objects. So we can find a common place to assert that both Frege and Weyl’s accounts take such expressions to stand for the *invariants* in question (*shapes*, *directions*, etc). Hence, the important point here is that those *invariants* are independent of us, that is, we cannot create them arbitrarily and their abstract norm of invariance is modelled in symbols.

6. Conclusion

In the preceding sections there was an attempt to highlight certain aspects of Weyl’s approach to Fregean implicit definitions. According to Frege, the content of an equivalence relation among the elements of an initial domain is reformulated as an identity statement where new expressions occur. Those expressions stand for abstract objects. On the other hand, Weyl’s account was construed in the context of his commitments to the tradition of phenomenology. Weyl describes the same procedure as a transformation of invariant features of the elements of an initial domain (that are involved in equivalence relations) into ideal objects. Weyl’s account puts emphasis in the role of intentionality as well as intuition in order to explain how the alleged objects come to be present in human mind. However, he makes justice to the symbolic construction of the procedure in a way that tends to safeguard objectivity.

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